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WAVE PROPAGATION OVER THE FREE SURFACE OF A TWO-PHASE MEDIUM WITH A NONUNIFORM CONCENTRATION OF THE DISPERSE PHASE

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The boundary-value problem of waves on the surface of a two-phase medium with a nonuniform (exponential) distribution of the disperse phase is formulated. An asymptotic solution of the linear problem in the form of damped progressive waves is obtained. The phase velocity, frequency, and damping decrement for the waves are found. The perturbation of the admixture concentration is determined, which, unlike in the case of a uniform distribution, is manifested even in a linear approximation. Numerical calculations were performed for concrete media.

Key words: *two-phase medium, wave, free surface, linear problem.*

Introduction. The linear problem of wave propagation over the free surface of a two-phase mixture with a uniform distribution of the disperse phase was solved in [1] and the nonlinear problem was solved in [2, 3]. These studies found the dependence of the wave parameters on the initial (unperturbed) admixture concentration, and its wave perturbation was obtained in [2, 3]. It was shown that in the case of a constant unperturbed concentration, its perturbation is a quantity of the higher order of smallness than the other quantities. The present paper deals with the formulation and solution of the problem of plane waves on the surface of a two-phase medium with an exponential distribution of the disperse-phase concentration over the depth. The multiple-velocity model of motion used in this case is more general than those employed in the above-mentioned studies because the equations of motion take into account not only interfacial friction but also the added-mass force.

1. Mathematical Model. We consider a two-phase mixture layer of constant depth on a horizontal solid substrate. From above, the layer is bounded by a free surface. The carrier phase ($i = 1$) is an ideal incompressible fluid, whose viscosity is manifested only at the interface, and the disperse phase ($i = 2$) consists of rigid particles. In the absence of heat and mass transfer, the motion of this medium is described by the following system of equations [4]:

$$\frac{\partial \rho_i}{\partial t^*} + \nabla(\rho_i \boldsymbol{v}_i^*) = 0,$$

$$\rho_i \frac{d\boldsymbol{v}_i^*}{dt^*} = -\alpha_i \nabla P_i + (-1)^i \alpha_1 \alpha_2 \Big[\frac{\rho_1^0}{2}\Big(\frac{d\boldsymbol{v}_1^*}{dt^*} - \frac{d\boldsymbol{v}_2^*}{dt^*}\Big) + R(\boldsymbol{v}_1^* - \boldsymbol{v}_2^*)\Big] + \rho_i \boldsymbol{g}, \tag{1.1}$$

$$\rho_i = \rho_i^0 \alpha_i, \qquad \alpha_1 + \alpha_2 = 1, \qquad \rho_i^0 = \text{const}, \qquad i = 1, 2.$$

Here α_i, $\boldsymbol{v}_i^*$, P_i, ρ_i, and ρ_i^0 are the volume concentration, velocity, pressure, and reduced and true densities of the ith phase, respectively, $\boldsymbol{g}$ is the acceleration of gravity, the coefficient R characterizes the Stokes viscous friction, e.g., for spherical particles of radius a, we have $R = 9\eta/(2a^2)$ [4], where η is the dynamic viscosity of the fluid; an asterisk denotes dimensional quantities (where necessary).

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We introduce Cartesian coordinates in such manner that the unperturbed free surface coincides with the plane $z^* = 0$ and the bottom with the plane $z^* = -l^*$ (l^* is the depth of the layer); the z^* axis is directed upward. We assume that in the absence of waves, the medium is at rest and the disperse phase is distributed exponentially over the depth of the mixture layer, i.e., $\alpha_2^0(z^*) = \alpha_0 \exp(\theta\delta^* z^*)$. Here α_0 is the particle concentration at the unperturbed surface $z^* = 0$; δ^* is a positive empirical coefficient that depends on the physical properties of the medium; the coefficient $\theta = 1$ for $\rho_2^0 < \rho_1^0$ or $\theta = -1$ at $\rho_2^0 > \rho_1^0$. Experimental observations of admixture distributions over the depth of quiescent mixtures [5] show that this model is fairly accurate. In order for system (1.1) to describe the wave motion of the mixture, it is necessary to introduce the wave perturbations of the concentration and pressure [1]:

$$\alpha_1 = 1 - \alpha_0 \exp(\theta\delta^* z^*) - \alpha', \qquad \alpha_2 = \alpha_0 \exp(\theta\delta^* z^*) + \alpha',$$
$$P_i = P_a - \rho_i^0 g z^* + p', \qquad i = 1, 2. \tag{1.2}$$

Here α' is the wave perturbation of the disperse-phase concentration and p' is the pressure perturbation. Substituting (1.2) into (1.1), we obtain the following system equations for the wave motion of the mixture:

$$-\frac{\partial\alpha'}{\partial t^*} + (1 - \alpha_0 \exp(\theta\delta^* z^*) - \alpha')\nabla \cdot \boldsymbol{v}_1^* - \alpha_0\theta\delta^* \exp(\theta\delta^* z^*) v_{1z}^* - \boldsymbol{v}_1^* \cdot \nabla\alpha' = 0,$$

$$\frac{\partial\alpha'}{\partial t^*} + (\alpha_0 \exp(\theta\delta^* z^*) + \alpha')\nabla \cdot \boldsymbol{v}_2^* + \alpha_0\theta\delta^* \exp(\theta\delta^* z^*) v_{2z}^* + \boldsymbol{v}_2^* \cdot \nabla\alpha' = 0,$$

$$\Big(\rho_1^0 + \frac{\rho_1^0}{2} (\alpha_0 \exp(\theta\delta^* z^*) + \alpha')\Big) \frac{d\boldsymbol{v}_1^*}{dt^*}$$
$$- \frac{\rho_1^0}{2} (\alpha_0 \exp(\theta\delta^* z^*) + \alpha') \frac{d\boldsymbol{v}_2^*}{dt^*} - R(\alpha_0 \exp(\theta\delta^* z^*) + \alpha')(\boldsymbol{v}_2^* - \boldsymbol{v}_1^*) + \nabla p' = 0, \tag{1.3}$$

$$\Big(\rho_2^0 + \frac{\rho_1^0}{2} (1 - \alpha_0 \exp(\theta\delta^* z^*) - \alpha')\Big) \frac{d\boldsymbol{v}_2^*}{dt^*}$$
$$- \frac{\rho_1^0}{2} (1 - \alpha_0 \exp(\theta\delta^* z^*) - \alpha') \frac{d\boldsymbol{v}_1^*}{dt^*} + R(1 - \alpha_0 \exp(\theta\delta^* z^*) - \alpha')(\boldsymbol{v}_2^* - \boldsymbol{v}_1^*) + \nabla p' = 0.$$

On the free surface $z^* = \xi(t^*, x^*, y^*)$, the following kinematic and dynamic boundary conditions are satisfied [1]:

$$\alpha_1 v_{1n}^* + \alpha_2 v_{2n}^* = V_n, \qquad P = \alpha_1 P_1 + \alpha_2 P_2 = P_a.$$

Here $\alpha_1 v_{1n}^* + \alpha_2 v_{2n}^*$ and V_n are the normal projections of the volume flow velocity of the mixture and the free surface. With allowance for (1.2), the boundary conditions for the case of plane–parallel wave motion become

$$\frac{\partial\xi}{\partial t^*} - (1 - \alpha_0 \exp(\theta\delta^*\xi) - \alpha')v_{1z}^* - (\alpha_0 \exp(\theta\delta^*\xi) + \alpha')v_{2z}^*$$
$$+ \frac{\partial\xi}{\partial x^*}\Big[(1 - \alpha_0 \exp(\theta\delta^*\xi) - \alpha')v_{1x}^* + (\alpha_0 \exp(\theta\delta^*\xi) + \alpha')v_{2x}^*\Big] = 0, \tag{1.4}$$
$$p - \Big[\rho_1^0(1 - \alpha_0 \exp(\theta\delta^*\xi) - \alpha') + \rho_2^0(\alpha_0 \exp(\theta\delta^*\xi) + \alpha')\Big] g\xi = 0, \qquad z^* = \xi(t^*, x^*).$$

Assuming that there is no mass flux through the solid surface of the horizontal substrate and the mixture slips along it, we obtain the following nonpenetration condition at the bottom ($z^* = -l^*$) [4]:

$$v_{iz}^* = 0, \qquad i = 1, 2. \tag{1.5}$$

Equations (1.3) and boundary conditions (1.4) and (1.5) constitute a mathematical model for the wave motion of the disperse mixture with a nonuniform distribution of the second phase in the layer at rest.

2. Formulation of the Boundary-Value Problem. Let a wave of length λ propagate over the free surface of the layer in the positive x^* direction. The wave length is much greater than the characteristic size of the disperse particles ($\lambda \gg a$). For progressive waves, it is assumed [6] that the solution can include the variable x^* only in the combination $x^* - c^*t^*$, where c^* is the phase velocity of the wave, which is to be determined. We introduce the following dimensionless variables and quantities:

$$t = kc^*t^*, \quad x = kx^*, \quad z = kz^*, \quad l = kl^*, \quad \mu_i = \rho_i^0/\rho^0, \quad r = R/(\rho^0 k c_0),$$
$$\delta = \delta^*/k, \quad \gamma = \alpha'/\alpha_0, \quad \zeta = k\xi, \quad \boldsymbol{v}_i = \boldsymbol{v}_i^*/c_0, \quad p = p'/(\rho^0 c_0^2), \quad c = c^*/c_0. \tag{2.1}$$

Here $\rho^0 = (1-\alpha_0)\rho_1^0 + \alpha_0\rho_2^0$ is the density of the quiescent mixture at the unperturbed free surface, c_0 is the phase velocity of the wave that corresponds to a uniform distribution of the admixture in the quiescent layer in a linear approximation, and $k = 2\pi/\lambda$ is the wave number. Substituting (2.1) into system (1.3)–(1.5), we obtain the following boundary-value problem:

$$-\alpha_0 c\,\frac{\partial\gamma}{\partial t} + (1-\alpha_0\exp(\theta\delta z) - \alpha_0\gamma)\Big(\frac{\partial v_{1x}}{\partial x} + \frac{\partial v_{1z}}{\partial z}\Big) - \alpha_0\theta\delta\exp(\theta\delta z)v_{1z} - \alpha_0\Big(v_{1x}\frac{\partial\gamma}{\partial x} + v_{1z}\frac{\partial\gamma}{\partial z}\Big) = 0,$$

$$\alpha_0 c\,\frac{\partial\gamma}{\partial t} + \alpha_0(\exp(\theta\delta z) + \gamma)\Big(\frac{\partial v_{2x}}{\partial x} + \frac{\partial v_{2z}}{\partial z}\Big) + \alpha_0\theta\delta\exp(\theta\delta z)v_{2z} + \alpha_0\Big(v_{2x}\frac{\partial\gamma}{\partial x} + v_{2z}\frac{\partial\gamma}{\partial z}\Big) = 0,$$

$$\Big(\mu_1 + \frac{\mu_1}{2}(\alpha_0\exp(\theta\delta z) + \alpha_0\gamma)\Big)c\,\frac{\partial v_{1s}}{\partial t} - \frac{\mu_1}{2}(\alpha_0\exp(\theta\delta z) + \alpha_0\gamma)c\,\frac{\partial v_{2s}}{\partial t}$$
$$- r(\alpha_0\exp(\theta\delta z) + \alpha_0\gamma)(v_{2s} - v_{1s}) + \frac{\partial p}{\partial s} - \frac{\mu_1}{2}(\alpha_0\exp(\theta\delta z) + \alpha_0\gamma)\Big(v_{2x}\frac{\partial v_{2s}}{\partial x} + v_{2z}\frac{\partial v_{2s}}{\partial z}\Big)$$
$$+\Big(\mu_1 + \frac{\mu_1}{2}(\alpha_0\exp(\theta\delta z) + \alpha_0\gamma)\Big)\Big(v_{1x}\frac{\partial v_{1s}}{\partial x} + v_{1z}\frac{\partial v_{1s}}{\partial z}\Big) = 0, \tag{2.2}$$
$$\Big(\mu_2 + \frac{\mu_1}{2}(1-\alpha_0\exp(\theta\delta z) - \alpha_0\gamma)\Big)c\,\frac{\partial v_{2s}}{\partial t} - \frac{\mu_1}{2}(1-\alpha_0\exp(\theta\delta z) - \alpha_0\gamma)c\,\frac{\partial v_{1s}}{\partial t}$$
$$+ r(1-\alpha_0\exp(\theta\delta z) - \alpha_0\gamma)(v_{2s} - v_{1s}) + \frac{\partial p}{\partial s} - \frac{\mu_1}{2}(1-\alpha_0\exp(\theta\delta z) - \alpha_0\gamma)\Big(v_{1x}\frac{\partial v_{1s}}{\partial x} + v_{1z}\frac{\partial v_{1s}}{\partial z}\Big)$$
$$+ \Big(\mu_2 + \frac{\mu_1}{2}(1-\alpha_0\exp(\theta\delta z) - \alpha_0\gamma)\Big)\Big(v_{2x}\frac{\partial v_{2s}}{\partial x} + v_{2z}\frac{\partial v_{2s}}{\partial z}\Big) = 0,$$
$$s = x, z.$$

The dimensionless kinematic and dynamic conditions at the free surface $z = \zeta(t, x)$ are written as follows:

$$c\,\frac{\partial\zeta}{\partial t} - (1-\alpha_0\exp(\theta\delta\zeta))v_{1z} - \alpha_0\exp(\theta\delta\zeta)v_{2z}$$
$$+ \frac{\partial\zeta}{\partial x}[(1-\alpha_0\exp(\theta\delta\zeta))v_{1x} + \alpha_0\exp(\theta\delta\zeta)v_{2x}]$$
$$- \alpha_0\gamma(v_{2z} - v_{1z}) + \alpha_0\gamma\,\frac{\partial\zeta}{\partial x}(v_{2x} - v_{1x}) = 0, \tag{2.3}$$
$$p - (\mu_1(1-\alpha_0\exp(\theta\delta\zeta)) + \mu_2\alpha_0\exp(\theta\delta\zeta))\nu_0^2\zeta + \alpha_0(\mu_1 - \mu_2)\nu_0^2\gamma\zeta = 0, \quad \nu_0^2 = g/(kc_0^2).$$

At the bottom ($z = -l$), we have

$$v_{iz} = 0, \qquad i = 1, 2. \tag{2.4}$$

The equations and boundary conditions (2.2)–(2.4) constitute a nonlinear boundary-value problem of determining the velocities of wave motion of the phases, the free-surface profile, and the pressure and concentration perturbations.

3. Solution of the Linear Problem. Let us consider a linear version of problem (2.2)–(2.4). We assume that the wave amplitude is smaller than the wavelength. Then, the kinematic and dynamic boundary conditions (2.3) specified at the unknown surface $z = \zeta(t, x)$ can be reduced to the conditions at the fixed surface $z = 0$. For this, all unknown functions included in them should be expanded in a Taylor series in the neighborhood of $z = 0$ [in particular, $\exp(\theta\delta\zeta) = 1 + \theta\delta\zeta + \delta^2\zeta^2/2 + \ldots$]. In addition, for surface waves of small amplitude, the velocities of wave motion of the phases and the wave perturbations are of the same order of magnitude as the quantity ζ, i.e., they are small [6]. Taking into account the smallness of the unknown quantities in (2.2)–(2.4), in Eqs. (2.2) and boundary conditions (2.3) expanded in a series in the neighborhood of $z = 0$, we retain only terms that are linear with respect to them. As a result, we have the linear problem

$$-\alpha_0 c\,\frac{\partial\gamma}{\partial t} + (1 - \alpha_0\exp(\theta\delta z))\Big(\frac{\partial v_{1x}}{\partial x} + \frac{\partial v_{1z}}{\partial z}\Big) - \alpha_0\theta\delta\exp(\theta\delta z)v_{1z} = 0,$$

$$c\,\frac{\partial\gamma}{\partial t} + \exp(\theta\delta z)\Big(\frac{\partial v_{2x}}{\partial x} + \frac{\partial v_{2z}}{\partial z}\Big) + \theta\delta\exp(\theta\delta z)v_{2z} = 0,$$

$$\Big(\mu_1 + \frac{\mu_1}{2}\,\alpha_0\exp(\theta\delta z)\Big)c\,\frac{\partial v_{1s}}{\partial t} - \frac{\mu_1}{2}\,\alpha_0\exp(\theta\delta z)c\,\frac{\partial v_{2s}}{\partial t} - r\alpha_0\exp(\theta\delta z)(v_{2x} - v_{1x}) + \frac{\partial p}{\partial s} = 0,$$

$$\Big(\mu_2 + \frac{\mu_1}{2}\,(1 - \alpha_0\exp(\theta\delta z))\Big)c\,\frac{\partial v_{2s}}{\partial t} - \frac{\mu_1}{2}\,(1 - \alpha_0\exp(\theta\delta z))c\,\frac{\partial v_{1s}}{\partial t} \qquad (3.1)$$

$$+\, r(1 - \alpha_0\exp(\theta\delta z))(v_{2s} - v_{1s}) + \frac{\partial p}{\partial s} = 0, \qquad s = x, z,$$

$$c\,\frac{\partial\zeta}{\partial t} = (1 - \alpha_0)v_{1z} + \alpha_0 v_{2z}, \qquad p - \nu_0^2\zeta = 0, \qquad \nu_0^2 = \frac{g}{kc_0^2}, \qquad z = 0.$$

The conditions at the bottom (2.4) remain unchanged.

Elimination of the unknown functions reduces system (3.1) to the problem for determining the pressure perturbation $p(t, x, z)$:

$$c^2\mu_1(\rho(z) + 2\mu_2)(\mu_1 + 2\mu(z))\,\frac{\partial^2}{\partial t^2}\,\Delta p + 4cr(\mu_1\rho(z) + \mu(z)\rho(z) + \mu_1\mu_2)\,\frac{\partial}{\partial t}\,\Delta p + 4r^2\rho(z)\Delta p$$

$$= \rho'(z)\Big[3c^2\mu_1(\mu_1 + 2\mu_2)\,\frac{\partial^3 p}{\partial t^2\,\partial z} + 4cr(2\mu_1 + \mu_2)\,\frac{\partial^2 p}{\partial t\,\partial z} + 4r^2\,\frac{\partial p}{\partial z}\Big],$$

$$\frac{\partial^3 p}{\partial t^3} + \frac{2r}{c\mu_1(1 + 2\mu_2)}\,\frac{\partial^2 p}{\partial t^2} + \frac{(\mu_1 + 2\mu^0)\nu_0^2}{c^2\mu_1(1 + 2\mu_2)}\,\frac{\partial^2 p}{\partial t\,\partial z} + \frac{2r\nu_0^2}{c^3\mu_1(1 + 2\mu_2)}\,\frac{\partial p}{\partial z} = 0, \qquad z = 0, \qquad (3.2)$$

$$\frac{\partial p}{\partial z} = 0, \qquad z = -l,$$

where

$$\rho(z) = \mu_1 + \alpha_0\exp(\theta\delta z)(\mu_2 - \mu_1), \qquad \mu(z) = \mu_2 - \alpha_0\exp(\theta\delta z)(\mu_2 - \mu_1),$$

$$\mu^0 = \mu(0), \qquad \Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2.$$

The unknown phase velocity of the wave c should be sought using the free-surface condition (3.2) in determining p. After that, the velocities of wave motion, the concentration perturbation, and the free-surface profile can be found from Eqs. (3.1).

The solution of the linear problem should meet a number of requirements. The relative motion of the phases leads to attenuation of the wave motion. In the absence of the disperse phase ($\alpha_0 = 0$) or in the case of identical true densities of the phases ($\rho_1^0 = \rho_2^0$), the solution of the problem becomes the well-known wave solutions for liquids [6]. Therefore, in the case of propagation of progressive waves over the free surface of the mixture, the solution of problem (3.2) should be sought in the form

$$p = \exp(-bt)[M(z)\sin(x - t) + N(z)\cos(x - t)]/\sinh l, \qquad (3.3)$$

where $b = \beta/(kc^*)$ is the dimensionless damping decrement (β is the dimensional decrement).

Substituting (3.3) into (3.2) and setting the coefficients at $\sin(x-t)$ and $\cos(x-t)$ equal to zero, we obtain a system of differential equations and boundary conditions. The equations for the unknown functions $M(z)$ and $N(z)$ have the form

$$
\begin{gathered}
[4r^2\rho(z) - 4bcr(\mu_1\rho(z) + \mu(z)\rho(z) + \mu_1\mu_2) \\
+ c^2(b^2-1)\mu_1(\rho(z) + 2\mu_2)(\mu_1 + 2\mu(z))](M''(z) - M(z)) \\
+ 2c[2r(\mu_1\rho(z) + \mu(z)\rho(z) + \mu_1\mu_2) - cb\mu_1(\rho(z) + 2\mu_2)(\mu_1 + 2\mu(z))](N''(z) - N(z)) \\
= [4r^2 - 4bcr(2\mu_1 + \mu_2) + 3c^2(b^2-1)\mu_1(\mu_1 + 2\mu_2)]\rho'(z)M'(z) \\
+ 2c[2r(2\mu_1 + \mu_2) - 3cb\mu_1(\mu_1 + 2\mu_2)]\rho'(z)N'(z), \\
2c[2r(\mu_1\rho(z) + \mu(z)\rho(z) + \mu_1\mu_2) - cb\mu_1(\rho(z) + 2\mu_2)(\mu_1 + 2\mu(z))](M''(z) - M(z)) \\
- [4r^2\rho(z) - 4bcr(\mu_1\rho(z) + \mu(z)\rho(z) + \mu_1\mu_2) \\
+ c^2(b^2-1)\mu_1(\rho(z) + 2\mu_2)(\mu_1 + 2\mu(z))](N''(z) - N(z)) \\
= 2c[2r(2\mu_1 + \mu_2) - 3cb\mu_1(\mu_1 + 2\mu_2)]\rho'(z)M'(z) \\
- [4r^2 - 4bcr(2\mu_1 + \mu_2) + 3c^2(b^2-1)\mu_1(\mu_1 + 2\mu_2)]\rho'(z)N'(z).
\end{gathered}
\tag{3.4}
$$

At $z = 0$, the following conditions should be satisfied:

$$
\begin{gathered}
M(z)[c^3b(3-b^2)\mu_1(1+2\mu_2) + 2c^2r(b^2-1)] + M'(z)\nu_0^2(2r - cb(\mu_1 + 2\mu^0)) \\
+ N(z)[c^3(3b^2-1)\mu_1(1+2\mu_2) - 4c^2br] + N'(z)\nu_0^2c(\mu_1 + 2\mu^0) = 0, \\
M(z)[c^3(3b^2-1)\mu_1(1+2\mu_2) - 4c^2br] + M'(z)\nu_0^2c(\mu_1 + 2\mu^0) \\
- N(z)[c^3b(3-b^2)\mu_1(1+2\mu_2) + 2c^2r(b^2-1)] - N'(z)\nu_0^2(2r - cb(\mu_1 + 2\mu^0)) = 0.
\end{gathered}
\tag{3.5}
$$

At $z = -l$, we have

$$
M'(z) = 0, \qquad N'(z) = 0. \tag{3.6}
$$

The solution of system (3.4) is sought in the form of series in a small parameter. As the small parameter we take δ, thus assuming that the admixture distribution in the quiescent layer is nearly uniform. The unknowns in (3.4)–(3.6) are sought in the form

$$
M(z) = \sum_{k=0}^{\infty} \delta^k M_k(z), \quad N(z) = \sum_{k=0}^{\infty} \delta^k N_k(z), \quad b = \sum_{k=0}^{\infty} \delta^k b_k, \quad c = 1 + \sum_{k=1}^{\infty} \delta^k c_k. \tag{3.7}
$$

Setting $\delta = 0$ in (3.4)–(3.6), we obtain the problem with a uniform admixture concentration in the quiescent layer [1]. Therefore, the solution of the problem in the zeroth approximation is written as

$$
\begin{gathered}
M_0(z) = K_0 \cosh\,(z + l), \qquad N_0(z) = L_0 \cosh\,(z + l), \\
c_0^2 = \frac{\mu_1 + 2\mu^0}{\mu_1(1+2\mu_2)}\,\nu_1^2 \tanh\, l + \tilde\beta\Big(3\tilde\beta - \frac{4r_1}{\mu_1(1+2\mu_2)}\Big), \\
\tilde\beta = \Big[-\frac{\chi}{2} + \sqrt{\frac{\chi^2}{4} + \frac{\psi^3}{27}}\,\Big]^{1/3} + \Big[-\frac{\chi}{2} - \sqrt{\frac{\chi^2}{4} + \frac{\psi^3}{27}}\,\Big]^{1/3} + \frac{2r_1}{3\mu_1(1+2\mu_2)}, \\
\psi = (3\mu_1(1+2\mu_2)(\mu_1 + 2\mu^0)\nu_1^2 \tanh\, l - 4r_1^2)/(12\mu_1^2(1+2\mu_2)^2), \\
\chi = r_1[4r_1^2 + 9\mu_1(1+2\mu_2)(\mu_1 - \mu + 3\mu_1\mu_2)\nu_1^2 \tanh\, l]/(54\mu_1^3(1+2\mu_2)^3), \\
r_1 = c_0 r = R/(\rho^0 k), \qquad \tilde\beta = c_0 b_0 = \beta_0/k, \qquad \nu_1^2 = c_0^2\nu_0^2 = g/k,
\end{gathered}
\tag{3.8}
$$

where K_0 and L_0 are arbitrary constants and β_0 is the dimensional damping decrement corresponding to the linear problem. Formulas (3.8) differ from those given in [1] by the terms due to the added-mass force. If this force is ignored in the initial equations (1.1), Eqs. (3.8) completely coincide with the results of [1].

To obtain equations and boundary conditions to find a first-approximation solution, we need to substitute (3.7) into (3.4)–(3.6). The small parameter δ appears explicitly in Eqs. (3.4); therefore, the quantities included in them should be previously expanded in a series in powers of δ. Substituting (3.7) into (3.4)–(3.6) and taking account of (3.8), for the unknown quantities $M_1(z)$ and $N_1(z)$, we obtain the following boundary-value problem:

$$\begin{gathered}
[4r^2 - 4b_0r(\mu_1 + \mu^0 + \mu_1\mu_2) + (b_0^2 - 1)\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)](M_1''(z) - M_1(z)) \\
+ 2[2r(\mu_1 + \mu^0 + \mu_1\mu_2) - b_0\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)](N_1''(z) - N_1(z)) \\
= \theta\alpha_0(\mu_2 - \mu_1)\{[4r^2 - 4b_0r(2\mu_1 + \mu_2) + (b_0^2 - 1)\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)]K_0 \\
+ 2[2r(2\mu_1 + \mu_2) - b_0\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)]L_0\} \sinh\,(z + l) = 0, \\
2[2r(\mu_1 + \mu^0 + \mu_1\mu_2) - b_0\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)](M_1''(z) - M_1(z)) \\
- [4r^2 - 4b_0r(\mu_1 + \mu^0 + \mu_1\mu_2) + (b_0^2 - 1)\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)](N_1''(z) - N_1(z)) \\
= \theta\alpha_0(\mu_2 - \mu_1)\{2[2r(2\mu_1 + \mu_2) - b_0\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)]K_0 \\
- [4r^2 - 4b_0r(2\mu_1 + \mu_2) + (b_0^2 - 1)\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)]L_0\} \sinh\,(z + l) = 0.
\end{gathered} \tag{3.9}$$

At $z = 0$, the following conditions should be satisfied:

$$\begin{gathered}
M_1(z)[b_0(3 - b_0^2)\mu_1(1 + 2\mu_2) + 2r(b_0^2 - 1)] + M_1'(z)\nu_0^2(2r - b_0(\mu_1 + 2\mu^0)) \\
+ N_1(z)[(3b_0^2 - 1)\mu_1(1 + 2\mu_2) - 4b_0r] + N_1'(z)\nu_0^2(\mu_1 + 2\mu^0) \\
+ b_1\{[(4b_0r - 3(b_0^2 - 1)\mu_1(1 + 2\mu_2))\cosh\,l - \nu_0^2(\mu_1 + 2\mu^0)\sinh\,l]K_0 \\
+ 2[3b_0\mu_1(1 + 2\mu_2) - 2r]\sinh\,lL_0\} \\
+ c_1\{[(4r(b_0^2 - 1) + 3b_0(3 - b_0^2)\mu_1(1 + 2\mu_2))\cosh\,l - \nu_0^2b_0(\mu_1 + 2\mu^0)\cosh\,l]K_0 \\
+ [(3(3b_0^2 - 1)\mu_1(1 + 2\mu_2) - 8rb_0)\cosh\,l + \nu_0^2(\mu_1 + 2\mu^0)\sinh\,l]L_0\} = 0, \qquad z = 0, \\
M_1(z)[(3b_0^2 - 1)\mu_1(1 + 2\mu_2) - 4b_0r] + M_1'(z)\nu_0^2(\mu_1 + 2\mu^0) \\
- N_1(z)[b_0(3 - b_0^2)\mu_1(1 + 2\mu_2) + 2r(b_0^2 - 1)] - N_1'(z)\nu_0^2(2r - b_0(\mu_1 + 2\mu^0)) \\
+ b_1\{2[3b_0\mu_1(1 + 2\mu_2) - 2r]\sinh\,lK_0 \\
- [(4b_0r - 3(b_0^2 - 1)\mu_1(1 + 2\mu_2))\cosh\,l - \nu_0^2(\mu_1 + 2\mu^0)\sinh\,l]L_0\} \\
+ c_1\{[(3(3b_0^2 - 1)\mu_1(1 + 2\mu_2) - 8rb_0)\cosh\,l + \nu_0^2(\mu_1 + 2\mu^0)\sinh\,l]K_0 \\
- [(4r(b_0^2 - 1) + 3b_0(3 - b_0^2)\mu_1(1 + 2\mu_2))\cosh\,l - \nu_0^2b_0(\mu_1 + 2\mu^0)\cosh\,l]L_0\} = 0, \qquad z = 0.
\end{gathered} \tag{3.10}$$

At the bottom, we have following conditions:

$$M_1'(z) = 0, \qquad N_1'(z) = 0, \qquad z = -l. \tag{3.11}$$

The solution of the system of linear inhomogeneous equations with constant coefficients (3.9) that satisfies the condition at the bottom (3.11) has the form

$$\begin{gathered}
M_1(z) = K_1\cosh\,(z + l) + (Y_1K_0 + Y_2L_0)(z\cosh\,(z + l) - \sinh\,(z + l)), \\
N_1(z) = L_1\cosh\,(z + l) + (-Y_2K_0 + Y_1L_0)(z\cosh\,(z + l) - \sinh\,(z + l)),
\end{gathered} \tag{3.12}$$

where the coefficients K_1 and L_1 are arbitrary and Y_1 and Y_2 are defined by the expressions

$$Y_1 = \theta\alpha_0(\mu_2 - \mu_1)\{[4r^2 - 4b_0 r(2\mu_1 + \mu_2) + 3(b_0^2 - 1)\mu_1(\mu_1 + 2\mu_2)]$$
$$\times [4r^2 - 4b_0 r(\mu_1 + \mu^0 + \mu_1\mu_2) + (b_0^2 - 1)\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)]$$
$$+ 4[2r(2\mu_1 + \mu_2) - 3b_0\mu_1(\mu_1 + 2\mu_2)][2r(\mu_1 + \mu^0 + \mu_1\mu_2) - b_0\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)]\}/(2D_1),$$
$$Y_2 = \theta\alpha_0(\mu_2 - \mu_1)\{[4r^2 - 4b_0 r(\mu_1 + \mu^0 + \mu_1\mu_2) + (b_0^2 - 1)\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)] \qquad (3.13)$$
$$\times [2r(2\mu_1 + \mu_2) - 3b_0\mu_1(\mu_1 + 2\mu_2)] - [4r^2 - 4b_0 r(2\mu_1 + \mu_2) + 3(b_0^2 - 1)\mu_1(\mu_1 + 2\mu_2)]$$
$$\times [2r(\mu_1 + \mu^0 + \mu_1\mu_2) - b_0\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)]\}/D_1,$$
$$D_1 = (4r^2 - 4b_0 r\mu_1(1 + 2\mu_2) + (b_0^2 + 1)\mu_1^2(1 + 2\mu_2)^2)(4r^2 - 4b_0 r(\mu_1 + 2\mu^0) + (b_0^2 + 1)(\mu_1 + 2\mu^0)^2).$$

As is known [7], in order for the expansion to be uniformly suitable, the ratios $M_n(z)/M_{n-1}(z)$ and $N_n(z)/N_{n-1}(z)$ should be limited over the entire range of z. In this case, this condition is satisfied if the condition $l < \delta^{-1}$ is valid. The coefficient δ can be expressed in terms of the admixture concentration at the surface (α_0) and at the bottom [$\alpha_l = \alpha(-l) = \alpha_0 \exp(-\theta\delta l)$]. From the last equality, we obtain $\delta = l^{-1} \ln(\alpha_0/\alpha_l)^\theta$. Then, the inequality $l < \delta^{-1}$ should be rewritten as $\ln(\alpha_0/\alpha_l)^\theta < 1$, whence we obtain the following constraints on the region of applicability of the model:

$$\alpha_0/\alpha_l < \mathrm{e} \quad \text{at} \quad \rho_2^0 < \rho_1^0, \qquad \alpha_l/\alpha_0 < \mathrm{e} \quad \text{at} \quad \rho_2^0 > \rho_1^0.$$

Thus, this model adequately describes the wave process if the disperse-phase concentration varies over the layer depth by a factor of less than e.

Substituting (3.12) into conditions (3.10) and taking into account the solution of the linear problem (3.7), we find that a nontrivial solution for K_0 and L_0 exists only if the following equalities hold:

$$b_1[(4b_0 r - 3(b_0^2 - 1)\mu_1(1 + 2\mu_2))\cosh l - \nu_0^2(\mu_1 + 2\mu^0)\sinh l]$$
$$+ c_1[(4r(b_0^2 - 1) + 3b_0(3 - b_0^2)\mu_1(1 + 2\mu_2))\cosh l - \nu_0^2 b_0(\mu_1 + 2\mu^0)\cosh l]$$
$$+ \nu_0^2 \tanh l[(2r - b_0(\mu_1 + 2\mu^0))Y_1 - (\mu_1 + 2\mu^0)Y_2] = 0,$$
$$2b_1[3b_0\mu_1(1 + 2\mu_2) - 2r] + c_1[(3(3b_0^2 - 1)\mu_1(1 + 2\mu_2) - 8rb_0)\coth l + \nu_0^2(\mu_1 + 2\mu^0)]$$
$$+ \nu_0^2 \tanh l[(\mu_1 + 2\mu^0)Y_1 + (2r - b_0(\mu_1 + 2\mu^0))Y_2] = 0.$$

From this system of linear inhomogeneous equations, we obtain b_1 and c_1:

$$b_1 = \nu_0^2\{Y_1[-\tanh^2 l(16b_0 r^2 + 2r[\mu_1 + 6\mu_1\mu_2 - 4\mu^0 - b_0^2(11\mu_1 + 18\mu_1\mu_2 + 4\mu^0)]$$
$$+ 6b_0(b_0^2 + 1)\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)) + 2\nu_0^2 r(\mu_1 + 2\mu^0)\tanh^3 l]$$
$$+ Y_2[-\tanh^2 l(8(b_0^2 - 1)r^2 + 2b_0 r[7\mu_1 + 18\mu_1\mu_2 - 4\mu^0 - b_0^2(5\mu_1 + 6\mu_1\mu_2 + 4\mu^0)]$$
$$+ 3(b_0^4 - 1)\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)) + \nu_0^2(2b_0 r - (b_0^2 + 1)(\mu_1 + 2\mu^0))(\mu_1 + 2\mu^0)\tanh^3 l]\}/D_2,$$
$$c_1 = \nu_0^2\{Y_1[\tanh^2 l(8r^2 - 12b_0 r(\mu_1 + 2\mu_2) + 3(b_0^2 + 1)(\mu_1^2 + 2\mu_1\mu^0 + 2\mu_1^2\mu_2 + 4\mu^0\mu_1\mu_2)) \qquad (3.14)$$
$$- \nu_0^2(\mu_1 + 2\mu^0)^2 \tanh^3 l] + Y_2[\tanh^2 l(8b_0 r^2 + 2r[\mu_1 + 6\mu_1\mu_2 - 4\mu^0 - b_0^2(5\mu_1 + 6\mu_1\mu_2 + 4\mu^0)]$$
$$+ 3b_0(b_0^2 + 1)\mu_1(1 + 2\mu_2)(\mu_1 + 2\mu^0)) - \nu_0^2(2r - b_0(\mu_1 + 2\mu^0))(\mu_1 + 2\mu^0)\tanh^3 l]\}/D_2,$$
$$D_2 = \nu_0^4(\mu_1 + 2\mu^0)^2 \tanh^2 l - 2(\mu_1 + 2\mu^0)[4b_0 r + 3(1 - b_0^2)\mu_1(1 + 2\mu_2)]\tanh l$$
$$+ (b_0^2 + 1)[16r^2 - 24b_0 r\mu_1(1 + 2\mu_2) + 9(b_0^2 + 1)\mu_1^2(1 + 2\mu_2)^2].$$

Thus, with accuracy up to δ^1, the dimensional damping decrement β and the phase velocity of the wave c^* are equal to

$$\beta = kc^*b = kc_0b_0 + \delta kc_0(b_1 + b_0c_1), \qquad c^* = c_0(1 + \delta c_1).$$

Substituting (3.8) and (3.12) into (3.7) and then into (3.3), we obtain the pressure perturbation

$$p = \exp(-bt)\{(K_0 \sin(x-t) + L_0 \cos(x-t)) \cosh(z+l)$$

$$+ \delta[(K_1 \sin(x-t) + L_1 \cos(x-t)) \cosh(z+l) \tag{3.15}$$

$$+ ([Y_1K_0 + Y_2L_0] \sin(x-t) + (-Y_2K_0 + Y_1L_0) \cos(x-t))(z \cosh(z+l) - \sinh(z+l))]\}/ \sinh l,$$

where the constants Y_1 and Y_2 are defined by formulas (3.13) and K_i and L_i can be obtained from additional initial data. The free-surface profile is determined with accuracy up to δ^1 from the condition (3.2):

$$\zeta = \coth l \exp(-bt)[K_0 \sin(x-t) + L_0 \cos(x-t)$$

$$+ \delta\{(K_1 - \tanh l(Y_1K_0 + Y_2L_0)) \sin(x-t) + (L_1 - \tanh l(-Y_2K_0 + Y_1L_0)) \cos(x-t)]\}/\nu_0^2.$$

At the initial time $t = 0$, if the wave crest passes through the z axis and the wave height is equal to L, then

$$K_0 = 0, \qquad L_0 = \nu_0^2 L \tanh l, \qquad K_1 = \nu_0^2 L^2 Y_2 \tanh^2 l, \qquad L_1 = \nu_0^2 L Y_1 \tanh^2 l. \tag{3.16}$$

Therefore, ζ can be rewritten as

$$\zeta = L \exp(-bt) \cos(x-t).$$

With allowance for (3.16), the expression for the pressure perturbation (3.15) takes the form

$$p = L\nu_0^2 \exp(-bt)\{\cos(x-t) \cosh(z+l) \tag{3.17}$$

$$+ \delta(Y_2 \sin(x-t) + Y_1 \cos(x-t))[(\tanh l + z) \cosh(z+l) - \sinh(z+l)]\}/ \cosh l.$$

Substituting (3.17) and the relation $c = 1 + \delta c_1$ into Eqs. (3.1), integrating them, and discarding terms higher than δ, we obtain the phase velocities and the concentration perturbation γ. The expressions defining the velocities are cumbersome but they are easy to derive. Therefore, we do not give them. The concentration perturbation have the form

$$\gamma = L \frac{\nu_0^2}{\cosh l} \frac{2\delta \exp(-bt)}{\alpha_0(\mu_2 - \mu_1)(b_0^2 + 1)^2} \sinh(z+l)$$

$$\times [(2b_0Y_1 + (b_0^2 - 1)Y_2) \sin(x-t) + ((b_0^2 - 1)Y_1 - 2b_0Y_2) \cos(x-t)]. \tag{3.18}$$

From (3.18) it follows that the concentration perturbation in the parameter δ is an order of magnitude smaller than the perturbation of the pressure and free-surface profile. However, unlike in the problem with a uniform concentration [1], the function γ is different from zero even in the first approximation in the amplitude parameter.

To illustrate the results obtained, we performed calculations for a disperse mixture of water with lighter and heavier particles for the following parameters: $l^* = 10$ m, $\lambda = 1$ m, $\rho_1^0 = 1000$ kg/m^3, $\eta = 1.004$ kg/(m $\cdot$ sec), $a = 0.25 \cdot 10^{-2}$ m, $\alpha_0 = 0.1$, and $\delta^* = 0.05$ 1/m. Then, the disperse-phase concentration in the quiescent layer $\alpha_2^0 = \alpha_0 \exp(0.05z^*)$ for $\rho_2^0 = 500$ kg/m^3 and $\alpha_2^0 = \alpha_0 \exp(-0.05z^*)$ for $\rho_2^0 = 1500$ kg/m^3. Figure 1a shows the perturbation of the disperse-phase concentration at the time $t = 0$ for a mixture of a liquid with particles of density $\rho_2^0 = 500$ kg/m^3. The perturbation has a wave nature and is periodic in x with a period equal to the wavelength. With increase in the depth, the perturbation damps and becomes virtually unnoticeable at a depth of 1 m ($z^* = -1$). The concentration perturbation damps rapidly with time (Fig. 1b). Similar curves for $\rho_2^0 = 1500$ kg/m^3 are presented in Fig. 2. A comparison of Fig. 1a and Fig. 2b shows that the wave propagation has a greater effect on the particle distribution in the case $\rho_2^0 < \rho_1^0$. However, as follows from a comparison of Fig. 1a and Fig. 2b, in this case, the perturbations of the disperse-phase concentration damp faster.

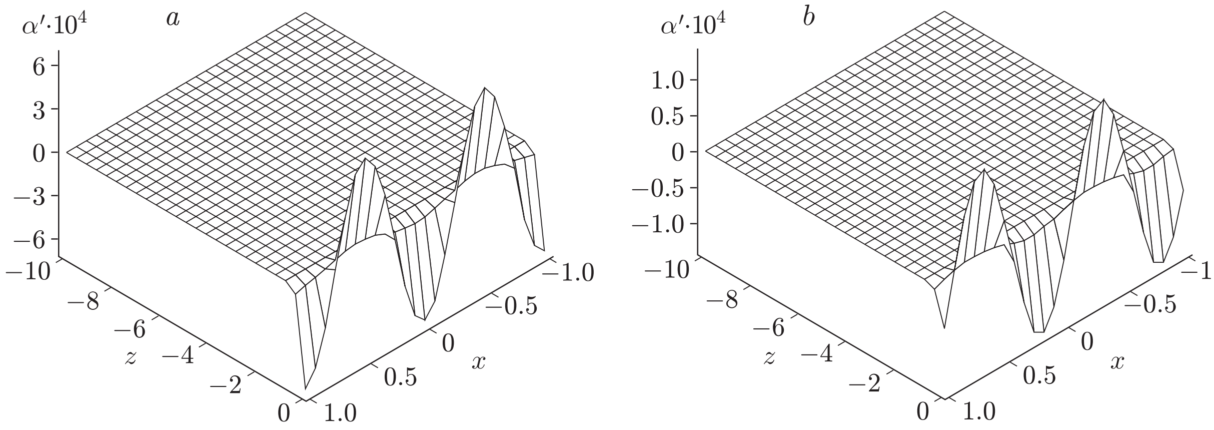

Fig. 1. Perturbation of the disperse-phase concentration for $\rho_2^0 = 500$ kg/m^3: $t = 0$ (a) and 300 sec (b).

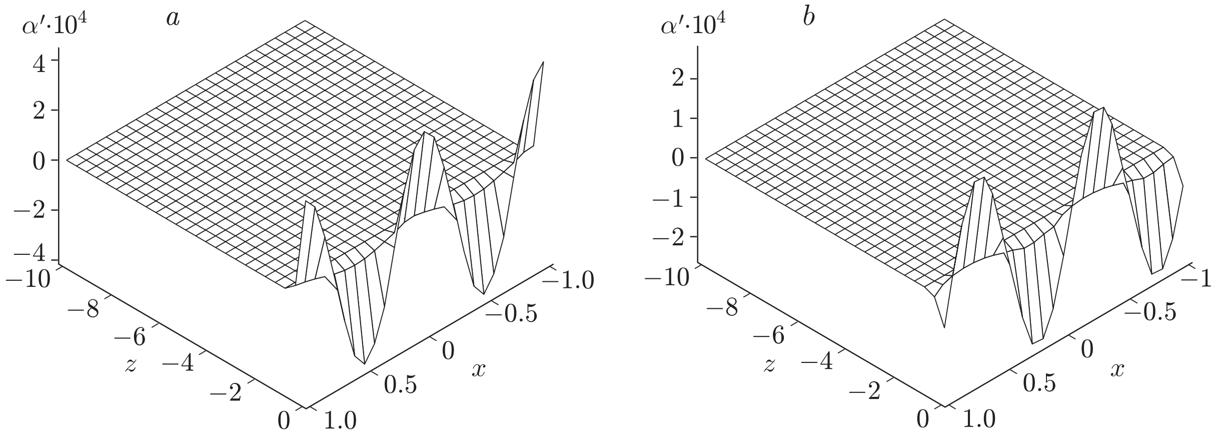

Fig. 2. Perturbation of the disperse-phase concentration for $\rho_2^0 = 1500$ kg/m^3: $t = 0$ (a) and 300 sec (b).

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